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SPECTRAL INEQUALITIES FOR POSITIVE SEMIDEFINITE HERMITIAN MATRICES

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Abstract. This paper investigates inequalities involving the largest eigenvalue, rank, and trace of positive semidefinite Hermitian matrices. The key estimates are established using the spectral theorem.
Keywords: Hermitian matrix, eigenvalue, matrix trace.

One of the most important classes in the theory of square matrices with complex entries is the class of Hermitian (or self-adjoint) matrices. Hermitian matrices play a fundamental role in spectral theory, quadratic forms, functional analysis, quantum mechanics, and optimization theory.

Definition 1. Let

$$A=\left(a_{ij}\right)\in M_n\left(\mathrm{C}\right)$$

be a complex square matrix. If

$$A=A^*$$

holds, then the matrix A is called a Hermitian matrix.

Here, A^* denotes the conjugate transpose of the matrix A, which is defined by

$$A^*=\overline{A}^T.$$

That is,

$$(A^*)_{ij}=\overline{a_{ji}}$$

Thus, the Hermitian condition can be written in terms of the entries as

$$a_{ij}=\overline{a_{ji}}, i,j=1,2,\dots,n.$$

Therefore, in a Hermitian matrix, the elements located symmetrically with respect to the main diagonal are complex conjugates of each other.

The diagonal elements of a Hermitian matrix consist of real numbers. Indeed, if $i=j$, then

$$a_{ii}=\overline{a_{ii}}$$

from which it follows that

$$a_{ii}\in\mathrm{R}.$$

Theorem 1. Suppose that

$$H\in\mathrm{C}^{n\times n}$$

is a Hermitian matrix, that is,

$$H^*=H.$$

Then the following statements hold:

i) All eigenvalues of H are real numbers;

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ii) There exists an orthonormal basis of eigenvectors of H in C^n;
iii) There exists a unitary matrix U such that

$$H = U\Lambda U^*,$$

where

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

is a diagonal matrix, and λ_i are the eigenvalues of H.
Moreover,

$$U^*U = UU^* = I,$$

that is, U is a unitary matrix.
A Hermitian matrix H is called positive semidefinite if, for every vector

$$x \in C^n,$$

the inequality

$$x^* H x \ge 0$$

is satisfied.
The above theorem is one of the fundamental results in the theory of Hermitian matrices.
Theorem 2. Let $H \in C^{n\times n}$ be a positive semidefinite Hermitian matrix, that is,

$$H = H^*, x^* H x \ge 0 \ \forall x \in C^n.$$

Then, for every $p \in N$,

$$\lambda_{\max}(H)^p \le \mathrm{tr}(H^p) \le \mathrm{rank}(H)\lambda_{\max}(H)^p.$$

Proof. Since H is a Hermitian matrix, by Theorem 1 (iii), H is unitarily diagonalizable. Therefore, there exist a unitary matrix U and real numbers $\lambda_1, \dots, \lambda_n$ such that

$$H = U\mathrm{diag}(\lambda_1, \dots, \lambda_n)U^*.$$

Here, $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the matrix H. Since H is positive semidefinite, all eigenvalues are nonnegative:

$$\lambda_i \ge 0, i = 1, \dots, n.$$

Indeed, if

$$Hv = \lambda v, \|v\| = 1,$$

then

$$\lambda = v^* H v \ge 0.$$

Now, for $p \in N$, the matrix H^p is determined by ordinary matrix multiplication. Since

$$U^*U = I,$$

we obtain

$$H^p = \left(U\mathrm{diag}(\lambda_1, \dots, \lambda_n)U^*\right)^p = U\mathrm{diag}(\lambda_1^p, \dots, \lambda_n^p)U^*.$$

Since the trace of a matrix is invariant under unitary similarity transformations,

$$\mathrm{tr}(H^p) = \lambda_1^p + \lambda_2^p + \dots + \lambda_n^p.$$

Now let

$$r = \mathrm{rank}(H).$$

For a positive semidefinite Hermitian matrix, the rank is equal to the number of nonzero eigenvalues. Therefore, the eigenvalues can be arranged in the following order:

$$\lambda_1,\dots,\lambda_r > 0, \lambda_{r+1} = \cdots = \lambda_n = 0.$$

Therefore,

$$\operatorname{tr}\left(H^p\right) = \sum_{i=1}^{r} \lambda_i^p.$$

Since the largest eigenvalue is

$$\lambda_{\max}\left(H\right) = \max_{1\le i\le r} \lambda_i,$$

for every $i = 1,\dots,r$, we have

$$0 < \lambda_i \le \lambda_{\max}\left(H\right).$$

Hence,

$$\lambda_i^p \le \lambda_{\max}(H)^p.$$

Now we estimate the sum:

$$\operatorname{tr}\left(H^p\right) = \sum_{i=1}^{r} \lambda_i^p \le \sum_{i=1}^{r} \lambda_{\max}(H)^p = r\,\lambda_{\max}(H)^p.$$

That is,

$$\operatorname{tr}\left(H^p\right) \le \operatorname{rank}\left(H\right)\lambda_{\max}(H)^p.$$

On the other hand, since $\lambda_{\max}\left(H\right)$ is one of the eigenvalues, the term

$$\lambda_{\max}(H)^p$$

appears in the sum. Because all terms are nonnegative,

$$\lambda_{\max}(H)^p \le \sum_{i=1}^{r} \lambda_i^p = \operatorname{tr}\left(H^p\right).$$

As a result,

$$\lambda_{\max}(H)^p \le \operatorname{tr}\left(H^p\right) \le \operatorname{rank}\left(H\right)\lambda_{\max}(H)^p.$$

The theorem is proved.

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